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Exact solutions of the motion equations for a classical spinning particle with radiation damping

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Abstract. A general solution of the Bhabha–Corben equations for a classical spinning particle with radiative damping is obtained. The solution only contains 'slow runaway terms' which are compatible with the principle of undetectability of small charges. Some conserved quantities are obtained which could be used to represent the total linear and angular momentum of the particle–field system. It is shown that the whole radiative effect may be renormalised in terms of 'effective variables' which describe a particle with constant mass in the frame of the Bhabha–Corben formulation without self-interaction radiative terms.

1. Introduction

Although several formulations (Bhabha and Corben 1941, Waysenhoff and Raabe 1947, Bargmann *et al* 1959, Halbwachs 1960) of the equations of motion for a relativistic classical spinning particle have been proposed, we shall only consider in this paper the Bhabha–Corben (1941) formulation which incorporates in a definite way the electromagnetic self-interaction. The original formulation contained explicit magnetic moment terms but it was argued later by Corben (1961) that the correct normal magnetic moment interaction was already contained in the equations with $g = 0$. Hence we shall concentrate on this case, which can then be considered to provide a complete description of a spinning particle in interaction with its own electromagnetic field.

There are two particular cases of this formulation which are of interest. The first is obtained by eliminating the spin variables of the particle. In this case the Bhabha–Corben equations reduce automatically to the Lorentz–Dirac equation for non-spinning particles with radiative damping. The general solution of these equations contains the famous 'runaway solutions' which are usually ruled out by postulating some appropriate boundary conditions (Rohrlich 1965). One may also get rid of these solutions if one introduces the principle of undetectability of small charges of Bhabha and Rohrlich (Rohrlich 1973). According to this principle, in the limit of vanishing charge the trajectory of the particle must have a limit which must coincide with the trajectory of the corresponding neutral particle with the same mass. If this physically sound principle is accepted, only the trivial motion $u_\mu = \text{constant}$ remains.

The second particular case may be obtained by dropping the non-linear Lorentz–Dirac radiative term in the Bhabha–Corben equations. This case has been considered extensively by Corben (1961, 1968) and the general solution is also known.

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However, exact solutions of the motion equations in the general case, including both spin variables and radiative effects, have not to the best of our knowledge yet been reported.

In this paper we show that the general Bhabha–Corben formulation contains two basic rates of evolution. The first corresponds to the runaway solutions of the Lorentz–Dirac equation. The second is found in the runaway solutions associated with the internal degrees of freedom introduced by the spin variables of the particle. It is a remarkable fact that, while the first type of runaway behaviour is again ruled out by the principle of undetectability of small charges, the second type is perfectly compatible with this principle. We then find the most general solution containing only runaway terms of the second type.

It is also shown that in this case the whole radiative effect may be renormalised by introducing 'effective variables' which obey the Bhabha–Corben equations without the self-interacting radiative term. The connection between the effective variables and the original ones may be constructed explicitly. It may also be seen that the renormalised mass is a constant and the kinematical conditions of the Bhabha–Corben formulation are verified. However, the interaction with an external electromagnetic field in terms of the effective variables cannot be minimal.

Since the equations of motion are invariant under the Poincaré group, a conserved total linear and angular momentum for the particle–field system should exist. In the process of finding the general solution of the motion equations, some conserved quantities are found explicitly which are natural candidates to play this role.

Recently Barut (1978, 1979) has proposed a model to understand the structure of the observed leptons as quantum excitations of the electron due to its radiative self-interaction. By considering iterative solutions of the Bhabha–Corben equations the anomalous magnetic moment of the classical particle is computed. The lepton masses are then obtained through some heuristic correspondence argument with the proposed quantum equation.

Even if the basic intuition contained in this work is correct, a better mathematical treatment of the classical equations is needed, both to confirm the calculation of the muon mass (Barut 1978) and to deal with leptons of higher mass (Barut 1979).

We think that the solutions obtained in the present work could be used as the starting point for a more rigorous discussion of the classical part of the argument. These solutions are also necessary as zero-order solutions in any perturbative approach to the classical equations in the presence of an external electromagnetic field.

The organisation of the paper is as follows. In § 2, the general Bhabha–Corben equations of motion are written in terms of normalised spin variables and the two fundamental rates of evolution of the problem are discussed. In § 3, the conserved quantities of the problem are found and used to exhibit the general solution of the motion equations and to discuss the renormalisation of the radiative effects.

2. Equations of motion and rates of evolution

We shall consider the description of a classical spinning particle with radiation damping given by the Bhabha–Corben (1941) equations

$$\frac{\mathrm{d}}{\mathrm{d}\tau}(mu_\mu + \dot{\sigma}_{\mu\nu}u^\nu) = \tfrac{2}{3}\alpha(\ddot{u}_\mu + \dot{u}^2 u_\mu) \tag{2.1}$$

$$\dot{\hat{\sigma}}_{\mu\nu} = \dot{\hat{\sigma}}_{\mu\lambda} u^\lambda u_\nu - \dot{\hat{\sigma}}_{\nu\lambda} u^\lambda u_\mu \tag{2.2}$$

where dots represent derivatives with respect to the proper time τ, m is the classical mass of the particle, u_μ its four-velocity, $\hat{\sigma}_{\mu\nu}$ the tensor variable associated with the classical spin and α the fine-structure constant. All through the paper the metric is taken to be $g_{\mu\nu} \equiv \text{diag}(1, -1, -1, -1)$ and the units are chosen to have $\hbar = c = 1$.

This system must be completed with the kinematical conditions:

$$u^\mu u_\mu = 1 \tag{2.3}$$

$$\hat{\sigma}_{\mu\nu} u^\nu = 0. \tag{2.4}$$

The equations (2.1)–(2.4) provide a complete description of a classical spinning particle in interaction with its own electromagnetic field.

2.1. *The spin vector*

Let us introduce the axial four-vector

$$\hat{S}_\mu \equiv \epsilon_{\mu\nu\lambda\rho} \hat{\sigma}^{\nu\lambda} u^\rho. \tag{2.1.1}$$

The spin tensor is given by the dual equation

$$\hat{\sigma}_{\mu\nu} = \tfrac{1}{2} \epsilon_{\mu\nu\lambda\rho} u^\lambda \hat{S}^\rho \tag{2.1.2}$$

and satisfies the condition (2.4) automatically.

The dynamical equation (2.2), written in terms of the spin vector, takes the simple form

$$u_\mu \dot{\hat{S}}_\nu - u_\nu \dot{\hat{S}}_\mu = 0. \tag{2.1.3}$$

Since $\hat{S}_\mu$ is orthogonal to u_μ from its definition (2.1.1), one obtains immediately

$$\hat{S}_\mu \dot{\hat{S}}^\mu = 0. \tag{2.1.4}$$

Hence, $\hat{S}_\mu$ is a spatial vector with constant norm:

$$\hat{S}_\mu \hat{S}^\mu = -\sigma^2 = \text{constant}. \tag{2.1.5}$$

We may then normalise the spin variables by introducing

$$S_\mu \equiv \hat{S}_\mu / \sigma \qquad \sigma_{\mu\nu} \equiv \hat{\sigma}_{\mu\nu} / \sigma. \tag{2.1.6}$$

Let us now divide equation (2.1) by σ and introduce for convenience

$$M \equiv m/\sigma \qquad \epsilon \equiv \tfrac{4}{3}\alpha/\sigma. \tag{2.1.7}$$

The first equation of motion may now be written as

$$\frac{\mathrm{d}}{\mathrm{d}\tau}(M u_\mu + \dot{\sigma}_{\mu\nu} u^\nu) = \tfrac{1}{2}\epsilon(\ddot{u}_\mu + \dot{u}^2 u_\mu). \tag{2.1.8}$$

The second equation of motion (2.2) is completely equivalent to (2.1.3), which states the proportionality between $\dot{S}_\mu$ and u_μ. Hence this equation may be written in the form

$$\dot{S}_\mu = \Lambda(\tau) u_\mu. \tag{2.1.9}$$

The system (2.1.8) and (2.1.9) is equivalent to the original dynamical system (2.1) and (2.2). The structure of the spin variables of the particle has been considerably simplified with the introduction of the unitary spatial spin vector S_μ.

2.2. Two rates of evolution of the system

Let us now consider two scalar quantities from which one may obtain the characteristic rates of evolution of the problem.

The first quantity is the function $\Lambda(\tau)$ introduced in (2.1.9):

$$\Lambda(\tau) \equiv \dot{S}_\mu u^\mu. \tag{2.2.1}$$

By contraction of the dynamical equation (2.1.8) with S_μ it is straightforward to obtain the equation

$$\dot{\Lambda}(\tau) = \frac{2M}{\epsilon}\Lambda(\tau) \tag{2.2.2}$$

which has the general solution

$$\Lambda(\tau) = \lambda \exp(\tau/\tau_0) \qquad \tau_0 \equiv \epsilon/2M. \tag{2.2.3}$$

This rate of evolution is characteristic of the 'runaway' solutions of the Lorentz–Dirac equation for non-spinning particles. It is evident that (2.2.3) is incompatible with the principle of undetectability of small charges of Bhabha and Rohrlich (Rohrlich 1973). According to this principle, in the limit $\alpha \to 0$, the trajectory of the particle must have a definite limit and this limit must be the trajectory of the neutral particle. This principle may be used in the Lorentz–Dirac equation to rule out the undesirable 'runaway' solutions. In the present case $\Lambda(\tau)$ diverges in this limit unless $\lambda = 0$.

The second scalar quantity of interest is the norm of the four-acceleration:

$$w^2(\tau) \equiv -\dot{u}_\mu \dot{u}^\mu. \tag{2.2.4}$$

By contraction of (2.1.8) with $\dot{u}_\mu$ it is easy to obtain

$$Mw^2(\tau) - \tfrac{1}{2}\epsilon_{\mu\nu\lambda\rho}\dot{u}^\mu u^\nu \ddot{u}^\lambda S^\rho = \tfrac{1}{2}\epsilon w(\tau)\dot{w}(\tau). \tag{2.2.5}$$

Using (2.1.8) again to eliminate $\ddot{u}_\mu$ one finds the equation

$$w(\tau)\dot{w}(\tau) = \frac{2M\epsilon}{1+\epsilon^2}w^2(\tau) + \frac{2M}{\epsilon(1+\epsilon^2)}\Lambda^2(\tau), \tag{2.2.6}$$

which has the general solution

$$\begin{aligned} w^2(\tau) &= \lambda^2 \exp(4M\tau/\epsilon) + \beta^2 \exp[4M\epsilon\tau/(1+\epsilon^2)] \\ &= \Lambda^2(\tau) + \beta^2 \exp(2\tau/\tau_0') \end{aligned} \tag{2.2.7}$$

with

$$\tau_0' \equiv (1+\epsilon^2)/2M\epsilon. \tag{2.2.8}$$

This is the second rate of evolution characteristic of the problem.

The two rates of evolution are essentially different in character. While the first type is ruled out by the principle of undetectability of small charges, the second type is perfectly compatible with it. In fact, for $\epsilon \to 0$, and taking $\lambda = 0$ one has

$$w^2 = \beta^2 = \text{constant}, \tag{2.2.9}$$

which is exactly the behaviour of the norm of the four-acceleration for a neutral spinning particle described by the Bhabha–Corben equations. This 'compatible runaway' is one of the most interesting characteristics of the problem.

3. Conserved quantities

The invariance of the dynamical equations under the Poincaré group suggests the existence of a conserved four-vector and of an antisymmetric tensor which could be interpreted as the total linear and angular momentum of the particle–field system. From a mathematical point of view, these conserved quantities could then be used as first integrals in order to obtain the general solution of the dynamical equations.

A most useful approach to the problem is obtained by introducing the dynamical tetrad

$$u_\mu,\ \dot{u}_\mu,\ p_\mu, S_\mu \tag{3.1}$$

where we have introduced the four-vector

$$p_\mu \equiv \dot{\sigma}_{\mu\nu} u^\nu. \tag{3.2}$$

The tetrad will be linearly independent if the invariant

$$I(\tau) \equiv \tfrac{1}{2}\epsilon_{\mu\nu\lambda\rho} u^\mu \dot{u}^\nu S^\lambda p^\rho = -p_\rho p^\rho \tag{3.3}$$

is different from zero for all τ.

It is a straightforward exercise to show that

$$I(\tau) = \tfrac{1}{4}(w^2(\tau) - \Lambda^2(\tau)) = \tfrac{1}{4}\beta^2 \exp(2\tau/\tau_0'). \tag{3.4}$$

Hence the tetrad will be independent for all τ if

$$\beta \neq 0. \tag{3.5}$$

In the case $\beta = 0$, the vector p_μ vanishes identically and the spin variables disappear in the dynamical equations. Hence only the well known 'runaway solutions' of the Lorentz–Dirac equation are left and those are incompatible with the principle of undetectability of small charges except for the trivial motion $u_\mu =$ constant. Accordingly we shall only consider the case $\beta \neq 0$ from now on.

3.1. Conserved four-vectors

Let us now look for a conserved four-vector C_μ which we represent in the dynamical tetrad

$$C_\mu(\tau) = T(\tau)u_\mu + \xi(\tau)\dot{u}_\mu + \eta(\tau)p_\mu + \zeta(\tau)S_\mu. \tag{3.1.1}$$

From the equations of motion (2.1.8) and (2.1.9) it is possible to obtain the useful relations

$$\dot{p}_\mu = \frac{M}{1+\epsilon^2}(-\dot{u}_\mu + \Lambda(\tau)S_\mu + 2\epsilon p_\mu) \tag{3.1.2}$$

$$\ddot{u}_\mu = w^2(\tau)u_\mu + \frac{2M\epsilon}{1+\epsilon^2}\dot{u}_\mu + \frac{2M\Lambda(\tau)}{\epsilon(1+\epsilon^2)}S_\mu + \frac{4M}{1+\epsilon^2}p_\mu \tag{3.1.3}$$

$$\dot{S}_\mu = \Lambda(\tau)u_\mu \tag{3.1.4}$$

and then it is easy to see that the conservation equations

$$\dot{C}_\mu(\tau) = 0 \tag{3.1.5}$$

are equivalent to the system

$$\dot{T}(\tau)+w^2(\tau)\xi(\tau)+\Lambda(\tau)\zeta(\tau)=0 \tag{3.1.6}$$

$$\dot{\xi}(\tau)+T(\tau)+\frac{2M\epsilon}{1+\epsilon^2}\xi(\tau)-\frac{M}{1+\epsilon^2}\eta(\tau)=0 \tag{3.1.7}$$

$$\dot{\eta}(\tau)+\frac{4M}{1+\epsilon^2}\xi(\tau)+\frac{2M\epsilon}{1+\epsilon^2}\eta(\tau)=0 \tag{3.1.8}$$

$$\dot{\zeta}(\tau)+\frac{2M\Lambda(\tau)}{\epsilon(1+\epsilon^2)}\xi(\tau)+\frac{M\Lambda(\tau)}{1+\epsilon^2}\eta(\tau)=0. \tag{3.1.9}$$

We shall only consider the case

$$\Lambda(\tau)\equiv 0. \tag{3.1.10}$$

As we have already discussed, this condition is necessary in order to rule out all the solutions with 'runaway' behaviour of the first type, incompatible with the principle of undetectability of small charges. Within this condition we are going to obtain the most general solution containing only 'slow runaway terms' compatible with this principle.

The condition (3.1.10) implies

$$\dot{S}_\mu=0 \qquad \dot{\zeta}(\tau)=0. \tag{3.1.11}$$

Hence the term ζS_μ in (3.1.1) is a constant which may then be absorbed in the conserved vector C_μ. Accordingly we shall consider the representation

$$C_\mu(\tau)=T(\tau)u_\mu+\xi(\tau)\dot{u}_\mu+\eta(\tau)p_\mu. \tag{3.1.12}$$

The conservation equations take now the simpler form

$$\begin{aligned}
&\dot{T}(\tau)+w^2(\tau)\xi(\tau)=0\\
&\dot{\xi}(\tau)+\frac{2M\epsilon}{1+\epsilon^2}\xi(\tau)+T(\tau)-\frac{M}{1+\epsilon^2}\eta(\tau)=0\\
&\dot{\eta}(\tau)+\frac{2M\epsilon}{1+\epsilon^2}\eta(\tau)+\frac{4M}{1+\epsilon^2}\xi(\tau)=0.
\end{aligned} \tag{3.1.13}$$

By introducing the variable

$$t\equiv\frac{1+\epsilon^2}{2M\epsilon}w(\tau)=\frac{1+\epsilon^2}{2\epsilon}\frac{\beta}{M}\exp[2M\epsilon\tau/(1+\epsilon^2)] \tag{3.1.14}$$

the system (3.1.13) may be separated into the third-order differential equation

$$t^2T'''(t)+tT''(t)+\left(\frac{1}{\epsilon^2}-t^2\right)T'(t)-tT(t)=0 \tag{3.1.15}$$

and the auxiliary equations

$$\xi(t)=-\frac{1+\epsilon^2}{2M\epsilon t}T'(t) \qquad \eta(t)=\frac{1+\epsilon^2}{M}(T(t)-T''(t)). \tag{3.1.16}$$

The general solution of (3.1.15) is a linear combination of the three independent solutions:

$$T_1(t) = 1 - \frac{\pi z}{\sin \pi\omega} J_\omega(z) J_{1-\omega}(z) \tag{3.1.17}$$

$$T_2(t) = \frac{\pi z}{2 \sin \pi\omega} (\exp(-\mathrm{i}\pi\omega) J_\omega(z) J_{\omega-1}(z) - \exp(\mathrm{i}\pi\omega) J_{-\omega}(z) J_{1-\omega}(z)) \tag{3.1.18}$$

$$T_3(t) = \frac{\pi z}{2 \sin \pi\omega} (\exp(-\mathrm{i}\pi\omega) J_\omega(z) J_{\omega-1}(z) + \exp(\mathrm{i}\pi\omega) J_{-\omega}(z) J_{1-\omega}(z)) \tag{3.1.19}$$

where the J's are Bessel functions of the first kind and

$$z \equiv \tfrac{1}{2}\mathrm{i}t \qquad \omega \equiv \tfrac{1}{2} + \mathrm{i}/2\epsilon. \tag{3.1.20}$$

The three functions verify the hyperbolic relation

$$T_1^2(t) - T_2^2(t) - T_3^2(t) = 1. \tag{3.1.21}$$

The normalisation has been chosen in such a way that the three corresponding conserved vectors are orthonormal:

$$C_{1\mu} C_1{}^\mu = 1 \qquad C_{2\mu} C_2{}^\mu = C_{3\mu} C_3{}^\mu = -1 \tag{3.1.22}$$

$$C_{1\mu} C_2{}^\mu = C_{1\mu} C_3{}^\mu = C_{2\mu} C_3{}^\mu = 0. \tag{3.1.23}$$

Of the three conserved vectors obtained, only the temporal one $C_{1\mu}$ has a limit when $t \to 0 (\tau \to -\infty)$. Hence we may think of some multiple of this vector as a natural candidate to represent the conserved total linear momentum of the particle–field system.

The classical motion of the particle is now completely determined. One may invert the three linear combinations of u_μ, $\dot{u}_\mu$ and p_μ equated to constants to obtain the expression for the four-velocity in terms of three constant orthonormal vectors which are determined from the initial conditions of the problem. Then the worldline $x^\mu(\tau)$ of the particle or any other physically interesting quantity may be obtained from this expression. It may be seen that a particle at rest for $t \to 0$ $(\tau \to -\infty)$ begins to self-accelerate, following an helical trajectory with increasing radius. This 'radial runaway' belongs to the 'slow regime' and disappears for $\alpha \to 0$.

It is a remarkable fact that the whole radiative effect may be renormalised in terms of 'effective variables'. In fact, one may introduce an effective position vector $X^\mu(\tau)$ and an effective spin tensor $\Sigma_{\mu\nu}(\tau)$ in such a way that the equations of motion (2.1) and (2.2) are completely equivalent to the equations

$$\frac{\mathrm{d}}{\mathrm{d}\tau}(M_\mathrm{R} U_\mu + \dot{\Sigma}_{\mu\nu} U^\nu) = 0 \tag{3.1.24}$$

$$\dot{\Sigma}_{\mu\nu} = \dot{\Sigma}_{\mu\lambda} U^\lambda U_\nu - \dot{\Sigma}_{\nu\lambda} U^\lambda U^\mu \tag{3.1.25}$$

where M_R is some constant mass and one has

$$U^\mu = \mathrm{d}X^\mu/\mathrm{d}\tau \qquad U^\mu U_\mu = 1 \qquad \Sigma_{\mu\nu} = \tfrac{1}{2}\epsilon_{\mu\nu\lambda\rho} U^\lambda S^\rho \tag{3.1.26}$$

where S_μ is a constant spatial spin vector. Equations (3.1.24) and (3.1.25) are identical to the Bhabha–Corben equations for a spinning particle without radiative damping.

The effective four-velocity U_μ may be constructed explicitly in terms of the 'old variables' and the three conserved vectors obtained. The whole construction is unique up to an affine transformation of the proper time. However, the interaction with an external electromagnetic field cannot be minimal when the problem is stated in terms of the effective variables.

A similar renormalisation procedure has been proposed by Corben (1962), but in general it requires a variable mass for the spinning particle and the kinematical condition $\Sigma_{\mu\nu}U^{\nu}=0$ does not hold for the effective variables.

3.2. Angular-momentum-type conserved quantities

One may also look for a conserved antisymmetric tensor $J_{\mu\nu}$ which could be used to represent the angular momentum of the system. For this purpose it is convenient to represent $J_{\mu\nu}$ in a basis of six independent antisymmetric tensors, namely

$$\begin{aligned}J_{\mu\nu}(\tau)=&A(\tau)(x_\mu u_\nu-x_\nu u_\mu)+B(\tau)(x_\mu\dot u_\nu-x_\nu\dot u_\mu)+C(\tau)(x_\mu p_\nu-x_\nu p_\mu)\\&+D(\tau)(u_\mu\dot u_\nu-u_\nu\dot u_\mu)+E(\tau)(u_\mu p_\nu-u_\nu p_\mu)+F(\tau)\sigma_{\mu\nu}.\end{aligned}\tag{3.2.1}$$

It may be seen that the conservation equations are now

$$\begin{aligned}&\dot A(\tau)+w^2(\tau)B(\tau)=0\\&\dot B(\tau)+\frac{2M\epsilon}{1-\epsilon^2}B(\tau)+A(\tau)-\frac{M}{1+\epsilon^2}C(\tau)=0\\&\dot C(\tau)+\frac{2M\epsilon}{1+\epsilon^2}C(\tau)+\frac{4M}{1+\epsilon^2}B(\tau)=0\end{aligned}\tag{3.2.2}$$

and

$$\begin{aligned}&\dot D(\tau)+\frac{2M\epsilon}{1+\epsilon^2}D(\tau)-\frac{M}{1+\epsilon^2}E(\tau)+B(\tau)=0\\&\dot E(\tau)+\frac{2M\epsilon}{1+\epsilon^2}E(\tau)+\frac{4M}{1+\epsilon^2}D(\tau)+C(\tau)-F(\tau)=0\\&\dot F(\tau)-w^2(\tau)E(\tau)=0.\end{aligned}\tag{3.2.3}$$

The system (3.2.2) is identical to (3.1.13) and accordingly its general solution may be written in terms of the functions $T^{(i)}(t)$ defined in (3.1.17)–(3.1.19).

Using now the variable t introduced in (3.1.14) the system (3.2.3) may be separated into the third-order differential equation

$$t^2F'''(t)+tF''(t)+\left(\frac{1}{\epsilon^2}-t^2\right)F'(t)-tF(t)=-\frac{2(1+\epsilon^2)}{M\epsilon^2}A'(t)\tag{3.2.4}$$

and the auxiliary equations

$$E(t)=\frac{1+\epsilon^2}{2M\epsilon t}F'(t)\qquad D(t)=\frac{1+\epsilon^2}{4M}(F(t)-F''(t)-C(t)).\tag{3.2.5}$$

It may be seen that the general solution of (3.2.4) is a linear combination of the three independent solutions

$$F^{(i)}(t)=T^{(i)}(t,\epsilon)-2\epsilon(1+\epsilon^2)\frac{\partial}{\partial\epsilon}T^{(i)}(t,\epsilon)\qquad i=1,2,3.\tag{3.2.6}$$

Hence we have obtained three independent conserved antisymmetric tensors with the structure

$$J^{(i)}_{\mu\nu}(t) = x_\mu C^{(i)}_\nu - x_\nu C^{(i)}_\mu + F^{(i)}(t)\sigma_{\mu\nu} + D^{(i)}(t)(u_\mu \dot{u}_\nu - u_\nu \dot{u}_\mu) + E^{(i)}(t)(u_\mu p_\nu - u_\nu p_\mu) \qquad i = 1, 2, 3. \tag{3.2.7}$$

Again, only $J^{(1)}_{\mu\nu}$ has a limit when $t \to 0$ ($\tau \to -\infty$), and is a natural candidate to represent the total angular momentum of the particle–field system.

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